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A LONGITUDINAL CRACK IN A PRESTRESSED THIN ELASTIC LAYER WITH FREE BOUNDARIES†

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The plane problem of a longitudinal crack loaded by a uniform pressure at its sides and symmetrically positioned in a prestressed thin layer with free boundaries is considered. The layer is prestressed in its plane by uniform forces applied at infinity. It is assumed that the material of a layer is described by an harmonic-type elastic potential. The additional stresses caused by the presence of the crack in the layer are considered to be small compared with the stresses of the main non-linear stress–strain state of the layer. This makes it possible to linearize the problem of determining the additional stresses on a background of the main stressed state. Such a linearized problem reduces to an integral equation of the first kind with a singular kernel with respect to a derivative of the function describing the crack opening. Asymptotic solutions of the integral equation for small values of the dimensionless parameter characterizing the layer thickness, are constructed for different values of the dimensionless parameter characterizing the prestressing of the layer. Examples are given.

Similar problems concerning cracks in prestressed bodies were examined earlier (see, for example, [1, 2]). The problem is studied here for the first time. © 2005 Elsevier Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM AND INITIAL RELATIONS

Consider an infinite elastic layer with an harmonic-type potential. In the initial state the layer is under conditions of a uniform field of normal stresses in its plane. There are no initial normal or shear stresses on areas parallel to the layer boundaries. With these assumptions, the displacements and stresses in the initial state are given by the formulae [3]

$$\begin{gathered} u_j^0 = (\lambda_j - 1)x_j = (\lambda_j - 1)\lambda_j^{-1}y_j, \quad \sigma_{jj}^0 = 2\mu\lambda_j^{-1}(\lambda_j - \lambda_2) \\ \lambda_j = \text{const}, \quad j = 1, 2, 3; \quad \lambda(\lambda_1 + \lambda_2 + \lambda_3 - 3) = -2\mu(\lambda_2 - 1) \end{gathered} \tag{1.1}$$

where x_j are Lagrangian coordinates, y_j are the Cartesian coordinates of the initial state, λ and μ are constants of elasticity, and λ_j are the coefficients of extension along the x_j axes – they are always positive. The region occupied by the layer in the initial state is defined by the conditions

$$|y_1| < \infty, \quad |y_2| \le h, \quad |y_3| < \infty$$

Suppose there is a crack in the middle plane of the layer that in the initial state occupies the region

$$|y_1| \le a, \quad y_2 = 0, \quad |y_3| < \infty$$

A uniform pressure q is applied to the sides of the crack, and the layer boundaries $y_2 = \pm h$ are load-free. We will assume that the disturbances of the initial stress field by the load q are small. In this case, the problem of determining the initial stresses and displacements can be linearized on the background of the main non-linear stress–strain state (1.1). Here, additional displacements u_1 and u_2 of points of the layer, caused by the load q, satisfy the following equilibrium equations describing plane strain [3]

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$$\alpha^2 b_1\frac{\partial^2 u_1}{\partial y_1^2}+b_2\frac{\partial^2 u_1}{\partial y_2^2}+\alpha\frac{\partial^2 u_2}{\partial y_1\partial y_2} = 0$$

$$b_1\frac{\partial^2 u_2}{\partial y_2^2}+\alpha^2 b_2\frac{\partial^2 u_2}{\partial y_1^2}+\alpha\frac{\partial^2 u_1}{\partial y_1\partial y_2} = 0 \tag{1.2}$$

$$b_1 = \frac{(1+\alpha)(\beta+2)}{(1+\alpha)\beta+2},\quad b_2 = \frac{2\alpha}{(1+\alpha)\beta+2};\quad \alpha = \frac{\lambda_1}{\lambda_2},\quad \beta = \frac{\lambda}{\mu}$$

while the additional stresses σ_{21} and σ_{22} are related to the additional displacements by the relations [3]

$$\sigma_{21} = \frac{2\mu}{\lambda_3(1+\alpha)}\left(\frac{\partial u_1}{\partial y_2}+\frac{\partial u_2}{\partial y_1}\right),\quad \sigma_{22} = \frac{\mu}{\lambda_3\alpha}\left(\alpha\beta\frac{\partial u_1}{\partial y_1}+(\beta+2)\frac{\partial u_2}{\partial y_2}\right) \tag{1.3}$$

(similar formulae for the additional stresses σ_{11} and σ_{12} are not required henceforth).

2. REDUCTION OF THE PROBLEM TO AN INTEGRAL EQUATION

We will first consider an auxiliary problem with the following boundary conditions

$$y_2 = h\colon \sigma_{21} = \sigma_{22} = 0$$

$$y_2 = 0\colon \sigma_{21} = 0,\quad \frac{\partial u_2}{\partial y_1} = \tilde{\gamma}'(y_1) \tag{2.1}$$

$$|y_1|\le a\colon \tilde{\gamma}'(y_1) = \gamma'(y_1);\quad |y_1|>a\colon \tilde{\gamma}'(y_1) = 0$$

To investigate it we will seek the solution of Eqs (1.2) in the form [3]

$$u_1 = -\frac{\partial^2\chi}{\partial y_1\partial y_2},\quad u_2 = \left(b_1\alpha\frac{\partial^2}{\partial y_1^2}+\frac{b_2}{\alpha}\frac{\partial^2}{\partial y_2^2}\right)\chi \tag{2.2}$$

Here, the first equation of (1.2) is satisfied identically, and the second leads to the following equation for the functions χ [3]

$$\left(\alpha^2\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right)^2\chi = 0 \tag{2.3}$$

Below, the function χ will be sought in the form of a Fourier integral

$$\chi = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\gamma, y_2)e^{-i\gamma y_1}d\gamma \tag{2.4}$$

Substituting expression (2.4) into Eq. (2.3), we obtain for the Fourier transform $X(\gamma, y_2)$, after solving an ordinary differential equation, the expression

$$X = [C_1(\gamma)+C_2(\gamma)\alpha|\gamma|y_2]e^{\alpha|\gamma|y_2}+[C_3(\gamma)+C_4(\gamma)\alpha|\gamma|y_2]e^{-\alpha|\gamma|y_2} \tag{2.5}$$

Now, to determine the quantities $C_l(\gamma)$ (l = 1, 2, 3, 4), we will represent the discontinuous function $\tilde{\gamma}'(y_1)$ of the form (2.1) as a Fourier integral

$$\tilde{\gamma}'(y_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\gamma)e^{-i\gamma y_1}d\gamma \tag{2.6}$$

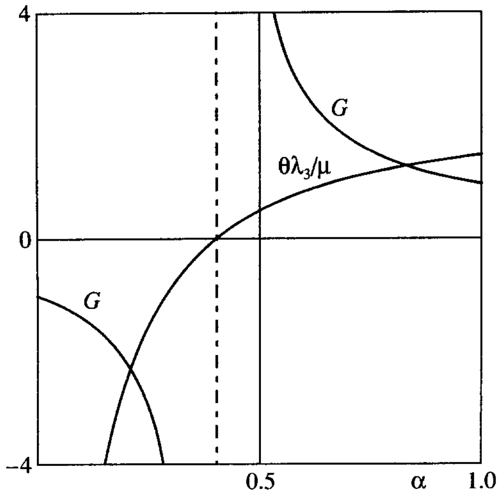

Fig. 1

and, using formulae (1.3), (2.2) and (2.4), rewrite boundary conditions (2.1) in terms of Fourier transforms. As a result, taking into account formula (2.5), we arrive at a system of four algebraic equations in the quantities $C_l(\gamma)$. Solving this system, we find, in particular,

$$\sigma_{22}(y_1, 0) = -\frac{i\theta}{2\pi}\int_{-\infty}^{\infty}\Gamma(\gamma)\frac{2(\operatorname{sh}^2\alpha h\gamma - \alpha^2h^2\gamma^2G^2)}{\operatorname{sh}2\alpha h\gamma + 2\alpha h\gamma G}e^{-i\gamma y_1}d\gamma$$
$$\theta = \mu\frac{(3\alpha-1)\beta+4\alpha-2}{\lambda_3\alpha(\beta+2)}, \quad G = \frac{(\alpha+1)\beta+2}{(3\alpha-1)\beta+4\alpha-2} \tag{2.7}$$

The quantity θ will be called the contact stiffness.

Figure 1 shows $\theta\lambda_3/\mu$ and G as a function of α for $\beta = 2$. It can be seen that, at a certain value $\alpha = \alpha_{cr} < 1$ (in the present case $\alpha_{cr} = 2/5$), the value of $\theta\lambda_3/\mu$ vanishes, while the quantity G goes to infinity. Below it will be assumed that $\alpha > \alpha_{cr}$.

We will now consider the main problem. Its boundary conditions differ from the boundary conditions of auxiliary problem (2.1) only in the fourth condition, which now has the form

$$y_2 = 0, \quad |y_1| \le a\colon \sigma_{22} = -q; \quad y_2 = 0, \quad |y_1| > a\colon \frac{\partial u_2}{\partial y_1} = 0 \tag{2.8}$$

where q is the uniform pressure acting on the sides of the crack.

Transforming relation (2.6), we obtain

$$\Gamma(\gamma) = \int_{-a}^{a}\gamma'(\xi)e^{i\gamma\xi}d\xi \tag{2.9}$$

Substituting expression (2.9) into relation (2.7) and then equating (2.7), when $|y_1| \le a$ according to the first condition of (2.8), to the quantity $-q$, we arrive at the following integral equation in the function $\gamma'(y_1)$

$$\int_{-a}^{a}\gamma'(\xi)K\left(\frac{\xi - y_1}{H}\right)d\xi = -\frac{\pi H}{\theta}q, \quad |y_1| \le a, \quad H = \alpha h$$
$$K(z) = \int_0^{\infty}L(u)\sin uz\,du, \quad L(u) = \frac{2(\operatorname{sh}^2u - u^2G^2)}{\operatorname{sh}2u + 2uG} \tag{2.10}$$

Note that for the function $L(u)$ the following asymptotic relations hold

$$\begin{aligned} L(u) &= 1 + O(e^{-2u}), \quad (u \to \infty) \\ L(u) &= (1-G)u - \frac{1-2G}{3(1+G)}u^3 + O(u^5) \quad (u \to 0) \end{aligned} \tag{2.11}$$

Below, we will use the integrals [4]

$$\int_0^\infty \sin uz du = \frac{1}{z}, \quad \int_0^\infty \cos uz du = \pi\delta(z) \tag{2.12}$$

where $\delta(z)$ is the data function. By virtue of the first relation of (2.11) and the first integral of (2.12), it follows that integral equation (2.10) is singular.

If $G = 1$ (this will be so when $\alpha = 1$) and $\lambda_3 = 1$, then, by virtue of the final formula of (1.1), $\lambda_1 = \lambda_2 = 1$. In this case, integral equation (2.10) becomes the integral equation of the problem of a crack in a non-prestressed layer with free boundaries. This problem was considered earlier [5]. Note also that in this particular case, by virtue of the second formula of (2.7),

$$\theta = 2\mu\frac{\lambda+\mu}{\lambda+2\mu} = \frac{\mu}{1-\nu} \tag{2.13}$$

where μ is the shear modulus and ν is Poisson's ratio.

3. THE DEGENERATE SOLUTION OF INTEGRAL EQUATION (2.10) AT $\alpha \neq 1$

Note that $\gamma'(y_1)$ is an odd function. We integrate integral equation (2.10) once with respect to y_1. We will have

$$\int_{-a}^{a} \gamma'(\xi) M\left(\frac{\xi - y_1}{H}\right) d\xi = -\frac{\pi}{\theta} q y_1, \quad |y_1| \le a \tag{3.1}$$

$$M(z) = \int_0^\infty \frac{L(u)}{u} \cos uz du \tag{3.2}$$

It can be shown that integral equations (2.10) and (3.1) are equivalent, and their common solution has the form [6]

$$\gamma'(y_1) = \frac{\omega(y_1)}{\sqrt{a^2 - y_1^2}} \tag{3.3}$$

where $\omega(y_1)$ is at least a continuous function.

It is well known [6] that the degenerate solution of integral equation (3.1) for low values of the parameter $\varepsilon = H/a$ will be determined if, in the kernel $M(z)$ of the form (3.2), the function $L(u)$ is replaced by the first term of its zero expansion. On the strength of this, according to the second formula of (2.11), and taking into account the second integral of (2.12), we will find

$$M(z) \sim (1-G)\pi\delta(z) \tag{3.4}$$

Substituting this expression into integral equation (3.1) and evaluating the integral using the well-known property of the delta function, we arrive at the following expression for the degenerate solution

$$\gamma'(y_1) = -\frac{q y_1}{(1-G)H\theta} \tag{3.5}$$

It has no characteristic root singularity at the point $x = \pm a$, as required by the form of (3.3), and therefore it is said to be degenerate.

Integrating equality (3.5) with respect to y_1 and using the condition of the crack closure

$$\gamma(\pm a) = 0 \tag{3.6}$$

we finally obtain

$$\gamma(y_1) = \frac{q(a^2 - y_1^2)}{2(1-G)H\theta} \tag{3.7}$$

Analysing formula (3.7), we see that, when $G > 1$ ($\alpha < 1$), the crack opening is negative. Physically this means that, for values $\alpha_{cr} < \alpha < 1$, the crack does not open. Below, we will examine the case when $\alpha \geq 1$.

From formula (3.7) it can be seen that, in the crack region, at very small values of the parameter ε, when we can confine ourselves to the degenerate solution, the layer is deformed in the same way as a membrane with restrained edges $y_1 = \pm a$.

4. A BOUNDARY-LAYER TYPE SOLUTION OF EQUATION (2.10) WHEN $\alpha > 1$

We will construct a boundary-layer type solution for small values of parameter ε in the neighbourhood of the points $y_1 = \pm a$. For this, in integral equation (3.1) we will replace the variables according to the formulae

$$t = \frac{a \pm y_1}{H}, \quad \tau = \frac{a \pm \xi}{H} \tag{4.1}$$

we will introduce the notation

$$\varphi_\pm(\tau) = \gamma'(\pm\tau H \mp a) \tag{4.2}$$

and we will let ε tend to zero in the upper limits of variation of τ and t. As a result we arrive at the following integral equation in the functions $\varphi_\pm(t)$

$$\int_0^\infty \varphi_\pm(\tau) M(\tau - t) d\tau = -\frac{\pi q}{H\theta}(\pm tH \mp a) \quad (0 \leq t < \infty) \tag{4.3}$$

Solutions of integral equation (4.3) can be found by the Wiener–Hopf method [7].

To construct solutions in analytical form, we will approximate the function $L(u)$ by the expression

$$L_*(u) = \frac{u\sqrt{u^2 + A^2}}{u^2 + B^2} \quad (A > 0, B > 0) \tag{4.4}$$

and we will examine an integral equation of type (4.3) with kernel

$$M_*(z) = \int_0^\infty \frac{L_*(u)}{u} \cos uz du \tag{4.5}$$

from which we find $\varphi_\pm^*(t)$.

It is well known [6] that, with this approach, the relative error

$$\sup_t |\varphi_\pm(t) - \varphi_\pm^*(t)| |\varphi_\pm(t)|^{-1} \tag{4.6}$$

will not exceed the relative error

$$\sup_u |L(u) - L_*(u)| |L(u)|^{-1} \tag{4.7}$$

and will have a smaller value the more accurately the function $L_*(u)$ approaches the function $L(u)$ for small values of u. Note that the function $L_*(u)$ tends to unity as $u \to \infty$, as required by the first relation of (2.11), and, to minimize the error of the solutions $\varphi^*_\pm(t)$, we will select approximation constants (4.4) so that, for $L_*(u)$, the second relation of (2.11) is satisfied. This will be so if

$$\frac{A}{B^2} = 1 - G, \quad \frac{2A^2 - B^2}{2AB^4} = \frac{1-2G}{3(1+G)} \tag{4.8}$$

We will introduce into consideration the Laplace–Carson originals of the functions

$$a_k(p) = \frac{1}{p^k\sqrt{p+A}}, \quad k = -2, -1, 0, 1, 2, 3 \tag{4.9}$$

We have [8]

$$\begin{aligned}
b_{-2}(t) &= -\frac{Ae^{-At}}{\sqrt{\pi t}} - \frac{e^{-At}}{2t\sqrt{\pi t}} + C, \quad b_{-1}(t) = \frac{e^{-At}}{\sqrt{\pi t}} \\
b_0(t) &= \frac{\operatorname{erf}(\sqrt{At})}{\sqrt{A}}, \quad b_1(t) = \frac{\operatorname{erf}(\sqrt{At})}{\sqrt{A}}\left(t - \frac{1}{2A}\right) + \frac{te^{-At}}{A\sqrt{\pi t}} \\
b_2(t) &= \frac{\operatorname{erf}(\sqrt{At})}{2\sqrt{A}}\left(t^2 - \frac{t}{A} + \frac{3}{4A^2}\right) + \frac{te^{-At}}{2A\sqrt{\pi t}}\left(t - \frac{3}{2A}\right) \\
b_3(t) &= \frac{\operatorname{erf}(\sqrt{At})}{6\sqrt{A}}\left(t^3 - \frac{3}{2A}t^2 + \frac{9}{4A^2}t - \frac{15}{8A^3}\right) + \frac{te^{-At}}{6A\sqrt{\pi t}}\left(t^2 - \frac{2}{A}t + \frac{15}{4A^2}\right)
\end{aligned} \tag{4.10}$$

where C is an infinite constant and erf(x) is the probability integral (all the formulae in (4.10) will be used in their entirety in Section 6). Now, as a result of using the Wiener–Hopf technique (we omit the details), we will obtain

$$\begin{aligned}
\varphi^*_\pm(t) &= -\frac{q}{H\theta}[\pm H\varphi_1(t) \mp a\varphi_0(t)] \\
\varphi_0(t) &= \frac{B}{\sqrt{A}}b_{-1}(t) + \frac{B^2}{\sqrt{A}}b_0(t) \\
\varphi_1(t) &= \frac{B-2A}{2A^{3/2}}b_{-1}(t) + \frac{B^2}{2A^{3/2}}b_0(t) + \frac{B^2}{\sqrt{A}}b_1(t)
\end{aligned} \tag{4.11}$$

It is easy to show that boundary-layer type solution (4.11) automatically increases exponentially with the degenerate solution (3.5). On the strength of this, the main term of the asymptotic form of the solution of integral equation (2.10) for low values of the parameter ε can be represented approximately in the form

$$\gamma'(y_1) \approx \varphi^*_+\left(\frac{a+y_1}{H}\right) + \varphi^*_-\left(\frac{a-y_1}{H}\right) + \frac{qy_1}{(1-G)H\theta} \tag{4.12}$$

Note that relation (4.12) already has the form of (3.2).

We will find the normal stress intensity factor at the crack tip (on its continuation) by means of formula (3.5) of [9]

$$N = -\lim \theta\gamma'(y_1)\sqrt{a-y_1} \quad (y_1 \to a) \tag{4.13}$$

On the basis of relations (4.11) and (4.12) we have

$$\frac{N}{q\sqrt{a}} \approx \frac{1}{2A\sqrt{\pi A\varepsilon}}[(-B+2A)\varepsilon + 2BA] \tag{4.14}$$

5. THE DEGENERATE SOLUTION OF INTEGRAL EQUATION (2.10) WHEN $\alpha = 1$

When $\alpha = 1$ we have $G = 1$, and the behaviour of the function $L(u)$ as $u \to 0$ changes qualitatively. Unlike the second formula of (2.11), we find

$$L(u) = \frac{1}{6}u^3 - \frac{1}{30}u^5 + O(u^7) \quad (u \to 0) \tag{5.1}$$

We integrate integral equation (2.10) three times with respect to y_1 and obtain

$$\begin{aligned} &\int_{-a}^{a} \gamma'(\xi) N\left(\frac{\xi - y_1}{H}\right) d\xi = \frac{\pi}{6H^2\theta}(qy_1^3 + D_* y_1), \quad |y_1| \le a \\ &N(z) = \int_0^\infty \frac{L(u)}{u^3} \cos uz du \end{aligned} \tag{5.2}$$

where D_* is still an arbitrary constant.

Integral equation (5.2) is equivalent to integral equation (2.10). Its general solution is given by the formula [10, 11]

$$\gamma'(y_1) = \frac{\Omega(y_1)}{(a^2 - y_1^2)^{3/2}} \tag{5.3}$$

where the function $\Omega(y_1)$ has at least a continuous first derivative. Note that the equivalence of these integral equations will be restored, and the structure of (5.3) will change to the structure of (3.2) if $\Omega(\pm a) = 0$. This can be achieved by an appropriate choice of the constant D_*, which will be done in Section 6.

As above, the degenerate solution of integral equation (5.2) for small values of parameter ε will be determined if in the kernel $N(z)$ of type (5.2) the function $L(u)$ is replaced by the first term of its expansion (5.1). By virtue of the above, and taking into account the second formula of (2.12), we obtain

$$N(z) \sim \pi\delta(z)/6 \tag{5.4}$$

Substituting expression (5.4) into integral equation (5.2) and evaluating the integral, we arrive at the following expression for the degenerate solution

$$\gamma'(y_1) = \frac{1}{H^3\theta}(qy_1^3 + D_* y_1) \tag{5.5}$$

As might have been expected, it does not possess the structure of (5.3).

In Section 6 it will be shown that for D_*, for small values of ε, the asymptotic formula

$$D_* = -qa^2[1 + O(\varepsilon)] \tag{5.6}$$

holds.

Substituting expression (5.6) into equality (5.5), neglecting the term $O(\varepsilon)$ and integrating with respect to y_1, taking into account the condition of the crack closure (3.6), we obtain

$$\gamma(y_1) = \frac{q}{4H^3\theta}(a^2 - y_1^2)^2 \tag{5.7}$$

From formula (5.7) it can be seen that, in the crack region, for very small values of the parameter ε, when it is possible to confine ourselves to the degenerate solution, the layer is deformed in the same way as a Kirchhoff–Love plate with clamped edges $y_1 = \pm a$.

6. BOUNDARY-LAYER TYPE SOLUTION OF EQUATION (2.10) WHEN $\alpha = 1$

As above, in integral equation (5.2) we will make the replacement of variables (4.1), we will introduce the notation (4.2) and we will let ε tend zero in the upper limits of the ranges of variation of τ and t. As a result, we will arrive at the following integral equation

$$\int_0^\infty \varphi_\pm(\tau)N(\tau - t)d\tau = \frac{\pi}{6H^3\theta}[q(\pm tH \mp a)^3 + D_*(\pm tH \mp a)] \quad (0 \le t < \infty) \tag{6.1}$$

The solutions of integral equation (6.1) can again be found by the Wiener–Hopf method [7].

To construct solutions in analytical form, we will approximate the function $L(u)$ by the expression

$$L_*(u) = \frac{u^3\sqrt{u^2 + A^2}}{(u^2 + B^2)^2} \quad (A > 0,\ B > 0) \tag{6.2}$$

and consider an integral equation of type (6.1) with kernel

$$N_*(z) = \int_0^\infty \frac{L_*(u)}{u^3}\cos uz du \tag{6.3}$$

from which we find $\varphi_\pm^*(t)$.

Note that the function $L_*(u)$ tends to unity as $u \to \infty$, as required by the first relation of (2.11), and we will select the approximation constants (6.2) such that for $L_*(u)$ relation (5.1) is satisfied. This will be so if

$$\frac{A}{B^4} = \frac{1}{6}, \quad \frac{4A^2 - B^2}{2B^6A} = \frac{1}{30} \tag{6.4}$$

We will introduce the notation

$$\kappa = B/A$$

and, using the Wiener–Hopf technique [7], we will obtain

$$\varphi_\pm^*(t) = \frac{1}{6H^3\theta}\{q[\pm H^3\varphi_3(t) \mp 3H^2a\varphi_2(t) \pm 3Ha^2\varphi_1(t) \mp a^3\varphi_0(t)] + D_*[\pm\varphi_1(t) \mp a\varphi_0(t)]\}$$

$$\varphi_0(t) = \frac{B^2}{\sqrt{A}}b_{-2}(t) + \frac{2B^3}{\sqrt{A}}b_{-1}(t) + \frac{B^4}{\sqrt{A}}b_0(t)$$

$$\varphi_1(t) = \frac{B}{\sqrt{A}}\left(-2 - \frac{1}{2}\kappa\right)b_{-2}(t) + \frac{B^2}{\sqrt{A}}(-3 + \kappa)b_{-1}(t) + \frac{B^4}{2A^{3/2}}b_0(t) + \frac{B^4}{\sqrt{A}}b_1(t)$$

$$\varphi_2(t) = \frac{1}{\sqrt{A}}\left(2 - 2\kappa + \frac{3}{4}\kappa^2\right)b_{-2}(t) + \frac{B^2}{A^{3/2}}\left(-3 + \frac{3}{2}\kappa\right)b_{-1}(t) + \tag{6.5}$$

$$+ \frac{B^2}{\sqrt{A}}\left(-4 + \frac{3}{4}\kappa^2\right)b_0(t) + \frac{B^4}{A^{3/2}}b_1(t) + \frac{2B^4}{\sqrt{A}}b_2(t)$$

$$\varphi_3(t) = \frac{1}{A^{3/2}}\left(3 - \frac{9}{2}\kappa + \frac{15}{8}\kappa^2\right)b_{-2}(t) + \frac{1}{\sqrt{A}}\left(6 - \frac{27}{4}\kappa^2 + \frac{15}{4}\kappa^3\right)b_{-1}(t) +$$

$$+ \frac{B^2}{A^{3/2}}(-6 + 15\kappa^2)b_0(t) + \frac{B^2}{\sqrt{A}}(-12 + 9\kappa^2)b_1(t) + \frac{3B^4}{A^{3/2}}b_2(t) + \frac{6B^4}{\sqrt{A}}b_3(t)$$

It can be shown that here a solution of the boundary-layer type (6.5) also increases automatically together with degenerate solution (5.5), but by a power law. On the strength of this, the main term of the asymptotic form of the solution of integral equation (2.10), for small values of the parameter ε, can be represented approximately in the form

$$\gamma'(y_1) \approx \varphi_+^*\left(\frac{a+y_1}{H}\right) + \varphi_-^*\left(\frac{a-y_1}{H}\right) - \frac{1}{H^3\theta}(qy_1^3 + D_* y_1) \tag{6.6}$$

Note that expression (6.6) has the form of formula (5.3). However, it is necessary for expression (6.6) to have the for of (3.2), and for this it is necessary in (6.5) to equate the coefficient of $b_{-2}(t)$ to zero. As a result of this, for D_* we obtain the expression

$$D_* = -\frac{qa^2 p_1(\varepsilon)}{4ABp_2(\varepsilon)} \tag{6.7}$$

$$p_1(\varepsilon) = (-24 + 36\kappa - 15\kappa^2)\varepsilon^3 + (48 - 48\kappa + 18\kappa^2)\varepsilon^2 +$$

$$+ BA(48 + 12\kappa)\varepsilon + 8B^2 A, \quad p_2(\varepsilon) = (4 + \kappa)\varepsilon + 2B$$

It can be seen that the expansion of D_* in powers of ε leads to asymptotic formula (5.6).

The normal stress intensity factor at the crack tip can be obtained from formula (4.13). On the basis of formulae (6.5)–(6.7) we have

$$\frac{N}{q\sqrt{a}} \approx \frac{p_3(\varepsilon)}{12\varepsilon p_2(\varepsilon)\sqrt{\pi A\varepsilon}} \tag{6.8}$$

$$p_3(\varepsilon) = (48 - 24\kappa + 12\kappa^2 - 24\kappa^3 + 15\kappa^4)\varepsilon^3 +$$

$$+ B(96 - 48\kappa + 42\kappa^2 - 18\kappa^3)\varepsilon^2 + B^2(48 - 12\kappa)\varepsilon + B^3(8 + 16\kappa)$$

7. EXAMPLES

Let $\beta = 2$. We will consider the cases when $\alpha = 2$ ($G = 1/2$) and $\alpha = 1$ ($G = 1$).

For approximation (4.4), using formulae (4.8), we obtain $A = 1$ and $B = \sqrt{2}$, and the error of the approximation will be 11%.

For approximation (6.2), using formulae (6.4), we obtain $A = 0.855878$ and $B = 1.505361$, and the error of the approximation will be 26%.

Figure 2 shows curves of $N/(q\sqrt{a})$ as a function of ε, calculated by means of formula (4.14) ($\alpha = 2$) and by means of formula (6.8) ($\alpha = 1$).

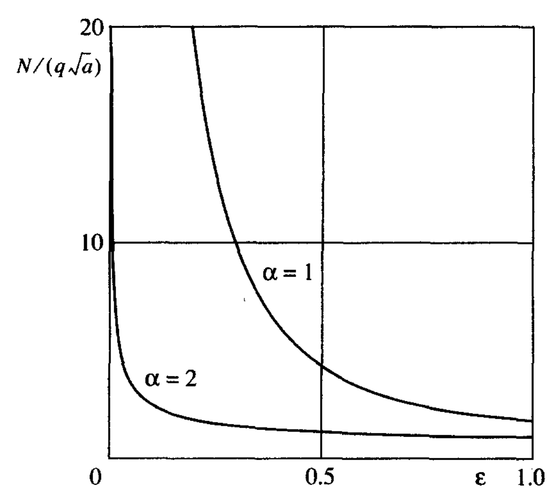

Fig. 2

The fracture criteria for prestressed bodies, using the normal stress intensity factor N at the crack tip, were given previously in [1].

CONCLUSIONS

From a comparison of curves 1 and 2 in Fig. 2 it can be concluded that the value of the parameter α, characterizing the prestressing of the layer, has a considerable effect on the magnitude of N.

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